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ABSTRACT

In this paper we study the relationships between the spaces of entire mappings of bounded type, entire mappings of nuclear bounded type, entire mappings of Pietsch integral bounded type, and entire mappings of Grothendieck integral bounded type. Several results due to Alencar (*Proc. Roy. Irish Acad.* **85A** (1985) 131–138) and Cilia and Gutiérrez (*J. Aust. Math. Soc.* **76** (2004) 269–280) for homogeneous polynomials are extended to entire mappings. In the main result we prove that an entire mapping is of nuclear bounded type if and only if it factors through an entire mapping of Pietsch integral bounded type.

INTRODUCTION

The spaces $\mathcal{H}_b(E;F)$ of entire mappings of bounded type and $\mathcal{H}_{Nb}(E;F)$ of entire mappings of nuclear bounded type have been studied by many authors. The aim of this paper is to introduce the spaces $\mathcal{H}_{PIb}(E;F)$ of entire mappings of Pietsch integral bounded type and $\mathcal{H}_{GIb}(E;F)$ of entire mappings of Grothendieck integral bounded type, and to establish the relationships between all these spaces.

The main result (Theorem 2.10) is a factorization theorem for entire mappings of nuclear bounded type which reads as follows: given $f \in \mathcal{H}_{Nb}(E;F)$, there are a separable reflexive Banach space R, a compact operator $T \in \mathcal{L}(E;R)$ and a mapping $g \in \mathcal{H}_{Nb}(R;F)$ such that $f = g \circ T$. Conversely, given a Banach space Z,

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a weakly compact operator $T \in \mathcal{L}(E; Z)$ and a mapping $g \in \mathcal{H}_{PIb}(Z; F)$, the mapping $f = g \circ T$ belongs to $\mathcal{H}_{Nb}(E; F)$. This result extends to entire mappings a result of Cilia and Gutiérrez [5, Theorem 7] for homogeneous polynomials.

1. NOTATION AND TERMINOLOGY

In this work, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0$ denotes the set $\mathbb{N} \cup \{0\}$. The letters E, X, Y, Z and F will always denote complex Banach spaces and E' represents the topological dual of E. B_E denote the closed unit ball of E. We denote by $c_0(E)$ the Banach space of all sequences $(x_n)_{n=1}^{\infty}$ of elements of E that converge to zero, with the supremum norm. We use $\mathcal{L}(E; F)$ for the space of all continuous operators from E into F. Given $m \in \mathbb{N}$, the space of all continuous m-homogeneous polynomials from E into F is denoted by $\mathcal{P}(^m E; F)$, with the supremum norm and for every $P \in \mathcal{P}(^m E; F)$ we can associate a unique symmetric m-linear (continuous) mapping $\check{P} : E \times \overset{(m)}{\cdots} \times E \to F$ such that $P(x) = \check{P}(x, \overset{(m)}{\ldots}, x)$ $(x \in E)$.

A polynomial $P \in \mathcal{P}(^m E; F)$ is said to be *nuclear* if it can be written in the form

$$P(x) = \sum_{j=1}^{\infty} \left[x'_j(x)\right]^m y_j \quad \text{for all } x \in E,$$

where $(x'_j) \subset E'$ and $(y_j) \subset F$ are bounded sequences such that $\sum_{j=1}^{\infty} \|x'_j\|^m \|y_j\| < \infty$. The space of all nuclear m-homogeneous polynomials from E into F is denoted by $\mathcal{P}_N(^m E; F)$, and is a Banach space for the norm

$$\|P\|_N := \inf\left\{\sum_{j=1}^{\infty} \left\|x'_j\right\|^m \|y_j\|\right\},$$

where the infimum is taken over all sequences $(x'_j) \subset E'$ and $(y_j) \subset F$ which satisfy the definition. We denote by $\mathcal{L}_N(E; F)$ the space of all nuclear operators from E into F.

A polynomial $P \in \mathcal{P}(^m E; F)$ is said to be *Pietsch integral* [2] if it can be written in the form

$$P(x) = \int_{B_{E'}} \left[x'(x)\right]^m dG\left(x'\right)$$

for all $x \in E$, where G is an F-valued regular countable additive Borel measure, of bounded variation, defined on $(B_{E'}$, weak-$*)$. The space of all Pietsch integral m-homogeneous polynomials from E into F is denoted by $\mathcal{P}_{PI}(^m E; F)$, and is a Banach space for the norm

$$\|P\|_{PI} := \inf |G|(B_{E'}),$$

where $|G|$ is the variation of G, and the infimum is taken over all measures satisfying the definition. We denote by $\mathcal{L}_{PI}(E; F)$ the space of all Pietsch integral operators from E into F. The definition of *Grothendieck integral* polynomials [4] is analogous, but taking the measure G to be F''-valued. The space of all Grothendieck integral polynomials is denoted by $\mathcal{P}_{GI}({}^m E; F)$. We denote by $\mathcal{L}_{GI}(E; F)$ the space of all Grothendieck integral operators from E into F.

$\mathcal{H}(E; F)$ will denote the vector space of all entire mappings from E into F. For each $f \in \mathcal{H}(E; F)$ we have its Taylor series in $a \in E$,

$$f(x) = \sum_{m=0}^{\infty} \frac{1}{m!} \hat{d}^m f(a)(x),$$

for every $x \in E$ and $\hat{d}^m f(a) \in \mathcal{P}({}^m E; F)$ for all $m \in \mathbb{N}_0$. A mapping $f \in \mathcal{H}(E; F)$ is said to be of *nuclear bounded type,* if

(i) $\frac{1}{m!}\hat{d}^m f(0) \in \mathcal{P}_N({}^m E; F)$, for all $m \in \mathbb{N}_0$.

(ii) $\lim_{m\to\infty}(\frac{1}{m!}\|\hat{d}^m f(0)\|_N)^{\frac{1}{m}} = 0$.

We denote by $\mathcal{H}_{Nb}(E; F)$ the space of all entire mappings of nuclear bounded type introduced by Gupta in [12,13]. For the general theory of polynomials and holomorphic mappings on Banach spaces, we refer the reader to the books of Dineen [10] and Mujica [14]. The definition of the Radon–Nikodým property may be found in [7, Definition III.1.3].

2. ENTIRE MAPPINGS OF PIETSCH INTEGRAL BOUNDED TYPE

Definition 2.1. A mapping $f \in \mathcal{H}(E; F)$ is said to be of *Pietsch integral bounded type* if:

(i) $\frac{1}{m!}\hat{d}^m f(0) \in \mathcal{P}_{PI}({}^m E; F)$, for all $m \in \mathbb{N}_0$.

(ii) $\lim_{m\to\infty}(\frac{1}{m!}\|\hat{d}^m f(0)\|_{PI})^{\frac{1}{m}} = 0$.

We denote by $\mathcal{H}_{PIb}(E; F)$ the vector space of all entire mappings of Pietsch integral bounded type from E into F.

In an analogous way we denote by $\mathcal{H}_{GIb}(E; F)$ the vector space of all entire mappings of Grothendieck integral bounded type from E into F. In [8] the authors characterize the functions in $\mathcal{H}_{PIb}(E; \mathbb{C}) = \mathcal{H}_{PIb}(E) = \mathcal{H}_{GIb}(E)$ by means of an integral representation on each ball in E. Since we do not use that integral representation, we refrain from writing the details, and refer the reader to the aforementioned paper.

The following result implies that $\mathcal{P}_{PI}({}^m E; F) \subset \mathcal{H}_{PIb}(E; F)$ for all $m \in \mathbb{N}$.

Lemma 2.2. *If $P \in \mathcal{P}_{PI}({}^m E; F)$, $k = 1, \ldots, m$, and $a \in E$, then $\hat{d}^k P(a) \in \mathcal{P}_{PI}({}^k E; F)$ and*

$$\left\| \frac{1}{k!}\hat{d}^k P(a) \right\|_{PI} \leqslant 2^m \|P\|_{PI} \|a\|^{m-k}.$$

Proof. It is analogous to the case $\mathcal{P}_{PI}({}^m E; \mathbb{C}) = \mathcal{P}_{PI}({}^m E)$, see Dineen [9, Proposition 3.19]. $\square$

The next result shows that in the definition of $\mathcal{H}_{PIb}(E;F)$ we can change the point 0, by any point $a \in E$ in conditions (i) and (ii).

Lemma 2.3. *Let $f \in \mathcal{H}_{PIb}(E;F)$. Then, for all $a \in E$,*

$$\lim_{m\to\infty}\left(\frac{1}{m!}\left\|\hat{d}^m f(a)\right\|_{PI}\right)^{\frac{1}{m}} = 0.$$

Therefore, the function $\tau_a(f) = f(a+\cdot)$ belongs to $\mathcal{H}_{PIb}(E;F)$.

Proof. It is analogous to the case $\mathcal{H}_{Nb}(E;F)$, see Gupta [12, Proposition 4.5]. $\square$

For each $m \in \mathbb{N}$, we can define in $\mathcal{H}_{PIb}(E;F)$ the seminorms

$$p_n(f) = \sum_{m=1}^{\infty} \frac{n^m}{m!}\left\|\hat{d}^m f(0)\right\|_{PI}.$$

It is easy to see that $(\mathcal{H}_{PIb}(E;F),(p_n)_n)$ is a Fréchet space. Moreover, for each $f \in \mathcal{H}_{PIb}(E;F)$, the sequence of partial sums of the Taylor series expansion of f about the origin converges to f in $\mathcal{H}_{PIb}(E;F)$.

Similar results are true, with simple modifications, for $\mathcal{P}_{GIb}(E;F)$ and $\mathcal{H}_{GIb}(E;F)$.

Every nuclear polynomial is Pietsch integral, and every Pietsch integral polynomial is Grothendieck integral. Moreover, if P is nuclear, we have

$$\|P\|_{GI} \leqslant \|P\|_{PI} \leqslant \|P\|_N.$$

Therefore $\mathcal{H}_{Nb}(E;F) \subset \mathcal{H}_{PIb}(E;F) \subset \mathcal{H}_{GIb}(E;F)$, and the inclusion mappings are continuous.

The following proposition extends results of Alencar [1,2].

Proposition 2.4. *Let E be a Banach space. Then the following conditions are equivalent:*

(i) *E' has the Radon–Nykodým property.*

(ii) *For every $m \in \mathbb{N}$ and every Banach space F, we have $\mathcal{P}_N({}^mE;F) = \mathcal{P}_{PI}({}^mE;F)$ with*

$$\|P\|_{PI} \leqslant \|P\|_N \leqslant \frac{m^m}{m!}\|P\|_{PI}$$

for all $P \in \mathcal{P}_{PI}({}^mE;F)$.

(iii) *For every $m \in \mathbb{N}$ and every Banach space F, we have $\mathcal{P}_N({}^mE;F) = \mathcal{P}_{PI}({}^mE;F)$ with*

$$\|P\|_{PI} \leqslant \|P\|_N \leqslant e^m\|P\|_{PI}$$

for all $P \in \mathcal{P}_{PI}({}^mE;F)$.

(iv) *For every Banach space F, we have $\mathcal{H}_{Nb}(E;F) = \mathcal{H}_{PIb}(E;F)$.*
(v) *For every Banach space F, we have $\mathcal{L}_N(E;F) = \mathcal{L}_{PI}(E;F)$.*

Proof. (i) $\Rightarrow$ (ii). See Alencar [2, Proposition 1].

(ii) $\Rightarrow$ (iii) Just observe that $\frac{m^m}{m!} \leqslant e^m$ for every $m \in \mathbb{N}$.

(iii) $\Rightarrow$ (iv). We know that $\mathcal{H}_{Nb}(E;F) \subset \mathcal{H}_{PIb}(E;F)$.

Let $f = \sum_{m=0}^{\infty} P_m \in \mathcal{H}_{PIb}(E;F)$, with $P_m = \frac{1}{m!}\hat{d}^m f(0) \in \mathcal{P}_{PI}(^m E;F)$ for all $m \in \mathbb{N}_0$. By (ii), we have that $P_m \in \mathcal{P}_N(^m E;F)$ for all $m \in \mathbb{N}$. Furthermore,

$$(\|P_m\|_N)^{\frac{1}{m}} \leqslant e \cdot (\|P_m\|_{PI})^{\frac{1}{m}} \overset{m\to\infty}{\longrightarrow} 0.$$

Thus $f \in \mathcal{H}_{Nb}(E;F)$.

(iv) $\Rightarrow$ (v). Let $T \in \mathcal{L}_{PI}(E;F)$. By Lemma 2.2, we have that, $T \in \mathcal{L}_{PI}(E;F) \subset \mathcal{H}_{PIb}(E;F) = \mathcal{H}_{Nb}(E;F)$. Thus $T \in \mathcal{L}_N(E;F)$.

(v) $\Leftrightarrow$ (i). See Alencar [1, Theorem 1.3]. In this case the identity mapping from $\mathcal{L}_N(E;F)$ to $\mathcal{L}_{PI}(E;F)$ is an isometry. $\square$

The following proposition extends a result of Cilia and Gutiérrez [5, Theorem 3].

Proposition 2.5. *If E' has the approximation property, then the following conditions are equivalent:*

(i) *E' has the Radon–Nykodým property.*
(ii) *For every $m \in \mathbb{N}$ and every Banach space F, we have $\mathcal{P}_N(^m E;F) = \mathcal{P}_{GI}(^m E;F)$ with*

$$\|P\|_{GI} \leqslant \|P\|_N \leqslant \frac{m^m}{m!}\|P\|_{GI}$$

for all $P \in \mathcal{P}_{GI}(^m E;F)$.
(iii) *For every $m \in \mathbb{N}$ and every Banach space F, we have $\mathcal{P}_N(^m E;F) = \mathcal{P}_{GI}(^m E;F)$ with*

$$\|P\|_{GI} \leqslant \|P\|_N \leqslant e^m\|P\|_{GI}$$

for all $P \in \mathcal{P}_{GI}(^m E;F)$.
(iv) *For every Banach space F, we have $\mathcal{H}_{Nb}(E;F) = \mathcal{H}_{GIb}(E;F)$.*
(v) *For every Banach space F, we have $\mathcal{L}_N(E;F) = \mathcal{L}_{GI}(E;F)$.*

Proof. (i) $\Rightarrow$ (ii). See Cilia and Gutiérrez [5, Theorem 3].

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v). Analogous to Proposition 2.4.

(i) $\Leftrightarrow$ (v). See Diestel [7, Theorem VIII.4.6]. $\square$

Lemma 2.6. *Let $f \in \mathcal{H}_b(E;F)$, $S \in \mathcal{L}(F;Y)$ and $T \in \mathcal{L}(X;E)$. If $f \in \mathcal{H}_{\Theta b}(E;F)$, then $S \circ f \circ T \in \mathcal{H}_{\Theta b}(X;Y)$, where Θ is N or PI or GI.*

Proof. Since $\hat{d}^m f(0) \in \mathcal{P}_\Theta({}^m E; F)$, $m \in \mathbb{N}$, we have that

$$\hat{d}^m (S \circ f \circ T)(0) = S \circ \hat{d}^m f(0) \circ T \in \mathcal{P}_\Theta\left({}^m X; Y\right),$$

Moreover, $\|S \circ \hat{d}^m f(0) \circ T\|_\Theta \leqslant \|S\| \cdot \|\hat{d}^m f(0)\|_\Theta \cdot \|T\|^m$.

Hence $S \circ f \circ T \in \mathcal{H}_{\Theta b}(X; Y)$. $\square$

The following proposition was proved by Cilia and Gutiérrez in [5, Proposition 5]. We give another proof of this proposition, for the estimates obtained in the proof will play a key role in the proof of the next proposition.

Proposition 2.7 [5]. *Let $S \in \mathcal{L}(F; Y)$ and $T \in \mathcal{L}(X; E)$ be weakly compact linear operators. There exist constants $C(S) > 0$ and $C(T) > 0$ such that for every $m \in \mathbb{N}$ and every $P \in \mathcal{P}_{GI}({}^m E; F)$, the following hold:*

(a) $S \circ P \in \mathcal{P}_{PI}({}^m E; Y)$ *and*

$$\|S \circ P\|_{PI} \leqslant C(S) \cdot \|P\|_{GI}.$$

(b) $J_F \circ P \circ T \in \mathcal{P}_N({}^m X; F'')$ *and*

$$\|J_F \circ P \circ T\|_N \leqslant C(T)^m \cdot \|P\|_{GI}.$$

Proof. (a) Since S is weakly compact, there are a reflexive Banach space R, and operators $A \in \mathcal{L}(F; R)$ and $B \in \mathcal{L}(R; Y)$ such that $S = B \circ A$ (see [6]). Since P is Grothendieck integral, $A \circ P \in \mathcal{P}_{GI}({}^m E; R)$. Since $R = R''$ it follows what $A \circ P \in \mathcal{P}_{PI}({}^m E; R)$ and $\|A \circ P\|_{PI} = \|A \circ P\|_{GI} \leqslant \|A\| \cdot \|P\|_{GI}$. Thus $S \circ P = B \circ A \circ P$ is Pietsch integral and

$$\|S \circ P\|_{PI} = \|B \circ A \circ P\|_{PI} \leqslant \|B\| \cdot \|A \circ P\|_{PI} \leqslant \|B\| \cdot \|A\| \cdot \|P\|_{GI}.$$

If we put $C(S) = \|B\| \cdot \|A\|$, then $\|S \circ P\|_{PI} \leqslant C(S) \cdot \|P\|_{GI}$.

(b) Since T is weakly compact, there are a reflexive Banach space R, and operators $A \in \mathcal{L}(X; R)$ and $B \in \mathcal{L}(R; E)$ such that $T = B \circ A$. Since P is Grothendieck integral, we have that $J_F \circ P \circ B \in \mathcal{P}_{GI}({}^m R; F'')$. Since F'' is complemented in its bidual, we have that $J_F \circ P \circ B$ is Pietsch integral. Since R is a reflexive Banach space, we have that R' has the Radon–Nykodým property (see [7, Corollary III.3.4]). Therefore $J_F \circ P \circ B$ is nuclear [3, Theorem 1.4] and consequently $J_F \circ P \circ T = J_F \circ P \circ B \circ A$ is nuclear. Moreover,

$$\begin{aligned}
\|J_F \circ P \circ T\|_N &\leqslant \|J_F \circ P \circ B\|_N \cdot \|A\|^m \quad \text{(by [3, Theorem 1.4])} \\
&= \|J_F \circ P \circ B\|_{PI} \cdot \|A\|^m \quad \text{(by [17])} \\
&= \|J_F \circ P \circ B\|_{GI} \cdot \|A\|^m \\
&\leqslant \|P \circ B\|_{GI} \cdot \|A\|^m \\
&\leqslant \|B\|^m \cdot \|A\|^m \cdot \|P\|_{GI}.
\end{aligned}$$

If we put $C(T) = \|B\| \cdot \|A\|$, then

$$\|J_F \circ P \circ T\|_N \leqslant C(T)^m \cdot \|P\|_{GI},$$

the result follows. $\square$

Next we extend the above result to the case of holomorphic mappings of bounded type.

Proposition 2.8. *Let $f \in \mathcal{H}_{GIb}(E; F)$ and let $S \in \mathcal{L}(F; Y)$ and $T \in \mathcal{L}(X; E)$ be weakly compact. Then:*

(a) $S \circ f \in \mathcal{H}_{PIb}(E; Y)$.
(b) $J_F \circ f \circ T \in \mathcal{H}_{Nb}(X; F'')$.

Proof. Let $f = \sum_{m=0}^{\infty} P_m \in \mathcal{H}_{GIb}(E; F)$, with $P_m = \frac{1}{m!}\hat{d}^m f(0) \in \mathcal{P}_{GI}(^m E; F)$ for all $m \in \mathbb{N}_0$ and $\lim_{m\to\infty} \|P_m\|_{GI}^{\frac{1}{m}} = 0$.

(a) Since S is weakly compact it follows from Proposition 2.7(a) that $S \circ P_m \in \mathcal{P}_{PI}(^m E; Y)$ for all $m \in \mathbb{N}$ and there is $C(S) > 0$ such that

$$\|S \circ P_m\|_{PI} \leqslant C(S) \cdot \|P_m\|_{GI}.$$

Thus

$$\lim_{m\to\infty} (\|S \circ P_m\|_{PI})^{\frac{1}{m}} \leqslant \lim_{m\to\infty} C(S)^{\frac{1}{m}} \cdot (\|P_m\|_{GI})^{\frac{1}{m}} = 0,$$

and we conclude that $S \circ f \in \mathcal{H}_{PIb}(E; Y)$.

(b) Since T is weakly compact, it follows from Proposition 2.7(b) that $J_F \circ P_m \circ T \in \mathcal{P}_N(^m X; F'')$ for all $m \in \mathbb{N}$ and there is $C(T) > 0$ such that

$$\begin{aligned}\lim_{m\to\infty} (\|J_F \circ P_m \circ T\|_N)^{\frac{1}{m}} &\leqslant \lim_{m\to\infty} \big[C(T)^m \cdot \|P_m\|_{GI}\big]^{\frac{1}{m}} \\ &= C(T) \cdot \lim_{m\to\infty} \|P_m\|_{GI}^{\frac{1}{m}} = 0.\end{aligned}$$

Hence $J_F \circ f \circ T \in \mathcal{H}_{Nb}(X; F'')$. $\square$

We need the following lemma to prove the main result of this paper. The proof of this lemma can be found in Pietsch [16, Lemma 8.6.4].

Lemma 2.9. *Let $(\sigma_j) \in \ell_1$. Then, given $\varepsilon > 0$, there exists $(\rho_j) \in c_0$ such that*

$$\sum_{j=1}^{\infty} \rho_j^{-2} |\sigma_j| \leqslant (1+\varepsilon) \sum_{j=1}^{\infty} |\sigma_j|$$

and $1 \geqslant \rho_1 \geqslant \rho_2 \geqslant \cdots > 0$.

Theorem 2.10. *Let $f \in \mathcal{H}(E;F)$. Then the following conditions are equivalent:*

(i) $f \in \mathcal{H}_{Nb}(E;F)$.

(ii) *f admits a factorization*

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "T"'] & & F \\ & Z \arrow[ur, "g"'] & \end{tikzcd}$$

where Z is a Banach space, $g \in \mathcal{H}_{Nb}(Z;F)$ and $T \in \mathcal{L}(E;Z)$ is a compact operator.

(iii) *f admits a factorization*

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "T"'] & & F \\ & R \arrow[ur, "g"'] & \end{tikzcd}$$

where R is a separable and reflexive Banach space, $g \in \mathcal{H}_{Nb}(R;F)$ and $T \in \mathcal{L}(E;R)$ is a compact operator.

(iv) *f admits a factorization*

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "T"'] & & F \\ & Z \arrow[ur, "g"'] & \end{tikzcd}$$

where Z is a Banach space, $g \in \mathcal{H}_{PIb}(Z;F)$ and $T \in \mathcal{L}(E;Z)$ is a weakly compact operator.

Proof. (i) $\Rightarrow$ (ii). Let $f \in \mathcal{H}_{Nb}(E;F)$. Let $\varepsilon > 0$. Since $P_m = \frac{1}{m!}\hat{d}^m f(0)$ is nuclear for every $m \in \mathbb{N}$, we can find a nuclear representation

$$P_m(x) = \sum_{j=1}^{\infty} \lambda_j^{(m)} \big[a_j^{(m)}(x)\big]^m y_j^{(m)} \quad (x \in E),$$

where $(a_j^{(m)}) \subset E'$, $(y_j^{(m)}) \subset F$ with $\|a_j^{(m)}\| = 1 = \|y_j^{(m)}\|$ for every $j, m \in \mathbb{N}$ and $(\lambda_j^m)_{j=1}^{\infty} \in \ell_1$ is such that

$$\sum_{j=1}^{\infty} \big|\lambda_j^{(m)}\big| < (1+\varepsilon)\|P_m\|_N.$$

By Lemma 2.9 for every m we can choose $(\rho_j^{(m)}) \in c_0$ with $1 \geqslant \rho_1^{(m)} \geqslant \rho_2^{(m)} \geqslant \cdots \geqslant \rho_j^{(m)} \geqslant \cdots > 0$ and

$$\sum_{j=1}^{\infty} \big[\rho_j^{(m)}\big]^{-2} \big|\lambda_j^{(m)}\big| < (1+\varepsilon)^2 \|P_m\|_N. \tag{2.1}$$

Now for each $m \in \mathbb{N}$, we define

$$\omega_j^{(m)} := \left[\rho_j^{(m)}\right]^{\frac{2}{m}}, \quad \text{for every } j \in \mathbb{N}.$$

Since $\rho_j^{(m)} \to 0$, then $\omega_j^{(m)} \to 0$. Moreover, $|\omega_j^{(m)}| \leqslant 1$ for all $j \in \mathbb{N}$. Now for each $m \in \mathbb{N}$, we define the operators $u_m : E \to c_0$ by

$$u_m(x) := \left(\omega_j^{(m)} a_j^{(m)}(x)\right)_{j=1}^{\infty}.$$

It is easy to see that for every $m \in \mathbb{N}$, u_m is a compact operator and $\|u_m\| \leqslant 1$.

Now, we consider the sequence $(\beta_m)_{m=1}^{\infty} = (\|P_m\|_N^{\frac{1}{2m}})_{m=1}^{\infty}$ and we define the operator $T : E \to c_0(c_0)$, by

$$T(x) := \left(\beta_m u_m(x)\right)_{m=1}^{\infty} \quad \text{for every } x \in E.$$

Thus, for all $x \in E$, $\|\beta_m u_m(x)\|_{c_0} \leqslant |\beta_m| \cdot \|x\|$. Since $\beta_m \to 0$, we have that $\beta_m u_m(x) \to 0$ in c_0. Clearly $T \in \mathcal{L}(E; c_0(c_0))$. If $i_m : c_0 \to c_0(c_0)$ denote the natural inclusion for every $m \in \mathbb{N}$, then $i_m \circ u_m : E \to c_0(c_0)$ is compact. Thus if we define

$$T_k(x) = \left(\beta_1 u_1(x), \ldots, \beta_k u_k(x), (0), (0), \ldots\right) = \sum_{j=1}^{k} \beta_j \cdot i_j \circ u_j(x)$$

for every $x \in E$ and $k \in \mathbb{N}$, then T_k is a compact operator and

$$\begin{aligned}
\|(T - T_k)(x)\|_{c_0(c_0)} &= \left\| \left(\beta_m u_m(x)\right)_{m=k+1}^{\infty} \right\|_{c_0(c_0)} \\
&= \sup_{m \geqslant k+1} \|\beta_m u_m(x)\|_{c_0} \\
&\leqslant \sup_{m \geqslant k+1} |\beta_m| \|u_m\| \|x\| \\
&\leqslant \Big(\sup_{m \geqslant k+1} |\beta_m|\Big) \|x\| \quad (\text{by } \|u_m\| \leqslant 1).
\end{aligned}$$

Since $\beta_m \to 0$, we have that $T_k \overset{k\to\infty}{\to} T$ and consequently T is a compact operator. For every $m \in \mathbb{N}$, put $\alpha_m = \frac{1}{\|P_m\|_N^{1/2}}$. Let $\pi_m : c_0(c_0) \to c_0$ and $p_j : c_0 \to \mathbb{C}$ denote the canonical projections. If we define $Q_m : c_0 \to F$ by

$$Q_m = \sum_{j=1}^{\infty} \alpha_m \left[\rho_j^{(m)}\right]^{-2} \lambda_j^{(m)} \cdot [p_j(\cdot)]^m \cdot y_j^{(m)}$$

then $Q_m \in \mathcal{P}_N({}^m c_0; F)$ and

$$\begin{aligned}
\|Q_m\|_N &\leqslant \sum_{j=1}^{\infty} \|p_j\|^m \alpha_m \cdot \left[\rho_j^{(m)}\right]^{-2} \cdot \left|\lambda_j^{(m)}\right| \cdot \left\|y_j^{(m)}\right\| \\
&= \alpha_m \sum_{j=1}^{\infty} \left[\rho_j^{(m)}\right]^{-2} \cdot \left|\lambda_j^{(m)}\right|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \alpha_m (1+\varepsilon)^2 \|P_m\|_N \quad \text{(by (2.1))}\\
&= (1+\varepsilon)^2 \|P_m\|_N^{1/2} \quad \text{(by definition of } \alpha_m).
\end{aligned}
$$

Thus $Q_m \circ \pi_m \in \mathcal{P}_N({}^m c_0(c_0); F)$ and

$$
\begin{aligned}
\lim_{m\to\infty} (\|Q_m \circ \pi_m\|_N)^{\frac{1}{m}} &\leqslant \lim_{m\to\infty} \big(\|Q_m\|_N \|\pi_m\|^m\big)^{\frac{1}{m}}\\
&\leqslant \lim_{m\to\infty} (\|Q_m\|_N)^{\frac{1}{m}}\\
&\leqslant \lim_{m\to\infty} \big[(1+\varepsilon)^2\big]^{\frac{1}{m}} (\|P_m\|_N)^{\frac{1}{2m}} = 0.
\end{aligned}
$$

If we define $g : c_0(c_0) \to F$ by $g = \sum_{m=1}^{\infty} Q_m \circ \pi_m$, then $g \in \mathcal{H}_{Nb}(c_0(c_0); F)$ and

$$
\begin{aligned}
g \circ T(x) &= \sum_{m=1}^{\infty} Q_m \circ \pi_m\big((\beta_m u_m(x))_{m=1}^{\infty}\big)\\
&= \sum_{m=1}^{\infty} Q_m\big(\beta_m u_m(x)\big)\\
&\qquad \text{(by definition of } \pi_m)\\
&= \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} \alpha_m \big[\rho_j^{(m)}\big]^{-2} \lambda_j^{(m)} \cdot \big[p_j\big(\beta_m u_m(x)\big)\big]^m \cdot y_j^{(m)}\\
&\qquad \text{(by definition of } Q_m)\\
&= \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} \big[\rho_j^{(m)}\big]^{-2} \cdot \big(\omega_j^{(m)}\big)^m \lambda_j^{(m)} \big[a_j^{(m)}(x)\big]^m \cdot y_j^{(m)}\\
&\qquad \text{(since } \alpha_m \cdot (\beta_m)^m = 1)\\
&= \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} \lambda_j^{(m)} \big[a_j^{(m)}(x)\big]^m \cdot y_j^{(m)}\\
&\qquad \text{(since } [\rho_j^{(m)}]^{-2} (\omega_j^{(m)})^m = 1)\\
&= \sum_{m=1}^{\infty} P_m(x) = f(x).
\end{aligned}
$$

(ii) $\Rightarrow$ (iii). Since $T : E \to Z$ is compact, there are a reflexive Banach space R and compact operators $A : E \to R$ and $B : R \to Z$ such what $T = B \circ A$ (see, e.g., [11, Corollary 3.3]). Since $\overline{A(B_E)}$ is a compact metric space, it is separable. Hence $R_1 = \overline{A(E)}$ is a separable and reflexive subspace of R and (iii) follows.

(iii) $\Rightarrow$ (iv). Clear.

(iv) $\Rightarrow$ (i). Let f as in (iv). Then we can find a reflexive space R and operators $A \in \mathcal{L}(E; R)$ and $B \in \mathcal{L}(R; Z)$ such that $T = B \circ A$. Since $g \in \mathcal{H}_{PIb}(Z; F)$, we have that $g \circ B : R \to F$ is Pietsch integral. Since R is a reflexive Banach space, we have that R' has the Radon–Nykodým property (see, e.g., [7, Corollary III.3.4]).

Then it follows from Proposition 2.4 that $g \circ B$ is nuclear. It is easy to see that $f = g \circ T = g \circ B \circ A : E \to F$ is nuclear. $\square$

Remark 2.11. According to Nachbin [15] and Dineen [9], a mapping $f \in \mathcal{H}(E)$ is said to be of nuclear holomorphy type at $\xi \in E$ if:

(1) $\hat{d}^m f(\xi) \in \mathcal{P}_N({}^m E)$, for all $m \in \mathbb{N}_0$.
(2) There are real numbers $C_1 \geqslant 0$, $C_2 \geqslant 0$ such that

$$\left\| \frac{1}{m!} \hat{d}^m f(\xi) \right\|_N \leqslant C_1 C_2^m \quad \text{for } m \in \mathbb{N}_0.$$

The mapping $f \in \mathcal{H}(E)$ is said to be of nuclear holomorphy type if f is of nuclear holomorphy type at all points of E. The space of all such mappings is denoted by $\mathcal{H}_N(E)$.

It is clear that $\mathcal{H}_{Nb}(E) \subseteq \mathcal{H}_N(E)$. On the other hand, $\mathcal{H}_{Nb}(E) \neq \mathcal{H}_N(E)$ in general (see, e.g., [9, Example 4.9]). Let us see that Theorem 2.10 does not hold in general for functions in $\mathcal{H}_N(E)$. Indeed, if $g \in \mathcal{H}(E; F)$ and $T \in \mathcal{L}(E; Z)$ is a compact operator, the composition $g \circ T \in \mathcal{H}_b(E; F)$. Hence mappings in $\mathcal{H}_N(E)$ do not factor through compact operators in general, otherwise the equality $\mathcal{H}_{Nb}(E) = \mathcal{H}_N(E)$ would hold true.

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